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NUMERICAL METHOD FOR UNSTEADY COMPRESSIBLE BOUNDARY LAYERS DRIVEN BY COMPRESSION OR EXPANSION WAVE

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SUMMARY

A numerical analysis is presented for the unsteady compressible laminar boundary layer driven by a compression or expansion wave. Approximate or series expansion methods have been used for the problems because of the characteristics of the governing equations, such as non-linearity, coupling with the thermal boundary layer equation and initial conditions. Here a transformation of the governing equations and the numerical linearization technique are introduced to deal with the difficulties. First, the governing equations are transformed for the initial conditions by Howarth and semisimilarity variables. These transformations reduce the number of independent variables from three to two and the governing equations from partial to ordinary differential equations at the initial point. Next, the numerical linearization technique is introduced for the non-linearity and the coupling with the thermal boundary layer equation. Because the non-linear terms are linearized without sacrifice of numerical accuracy, the solutions can be obtained without numerical iterations. Therefore the exact numerical solution, not approximate or series expansion, can be obtained. Compared with the approximate or series expansion method, this method is much improved. Results are compared with the series expansion solutions.

KEY WORDS Unsteady laminar compressible boundary layer Non-iterative finite difference method Semisimilarity transformation

INTRODUCTION

Numerical solutions are obtained for the unsteady compressible laminar boundary layer driven by a centred expansion or compression wave. The expansion wave type of flow occurs in the driver section of a conventional shock tube after the diaphragm brusts. A similar expansion wave flow appears in the reservoir tube of a Ludvig- or tube-type wind tunnel. The compression wave type of flow might be generated by an accelerated piston such as in isentropic compression tubes.

Inviscid expansion or compression wave flow has been solved by many investigators for one-dimensional unsteady isentropic flow in a shock tube. For example, Huter *et al.*[1] presented analytical solutions for velocity, pressure and temperature.

Several methods have been investigated to solve the unsteady compressible laminar boundary layer equations for viscosity effects. However, the solutions have been obtained by approximate or series expansion methods because of the characteristics of non-linearity, coupling with the

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thermal boundary layer equation and initial conditions. For instance, Cohen[2] used a co-ordinate expansion method to solve expansion wave flow. Hall[3] considered the expansion and the compression wave with a co-ordinate expansion method. Chang and Chen[4] also solved the expansion wave using a co-ordinate expansion.

This paper presents a method to obtain exact numerical solutions, not approximate or series expansion. Because the governing equations are partial parabolic differential, temporal and spatial initial conditions are required. For oscillatory or transition flows the temporal initial condition is supplied from outside the method, e.g. from physical observations or by given physical constraints, and the spatial initial conditions is given by variable transformation. However, both these initial conditions must be obtained from inside the method by variable transformations for an initially developed boundary layer flow. If compatible transformation variables are not found, the solutions can only be calculated by approximate or series expansion methods. Here the governing equations are simplified for the initial conditions by Howarth and semisimilarity variables, which can reduce the number of independent variables from three to two and the governing equations from partial to ordinary differential equations at the initial point.

In general, iterative numerical methods have been widely used for non-linear and coupled equations. These iterative methods are very inefficient for the characteristics of stability, convergence and computer time. In particular, coupled problems with the thermal boundary layer equation are more difficult.

Here the non-linear terms are linearized without sacrifice of numerical accuracy by the linearization technique developed by Orlandi and Ferziger[5] and Kim and Chang.[6,7]

The transformed equations are written in a system of five first-order partial differential equations, and the first-order equations are discretized and linearized by incremental variables and the linearization technique. These linearized implicit finite difference equations produce a 5×5 block tridiagonal matrix equation when assembled, which is inverted by a block elimination method. Therefore the solutions can be obtained without iterations.

Results are presented for two wall conditions, i.e. isothermal or adiabatic walls, and compared with the solutions of the series expansion method for expansion wave flow.

GOVERNING EQUATIONS AND TRANSFORMATION

The unsteady compressible laminar boundary layer flows are considered for an expansion or compression wave which travels into a stationary gas with a wavehead velocity of constant sound speed. The external flow of the boundary layer is assumed to be one-dimensional unsteady isentropic flow, which is well known.[1] In a co-ordinate system attached to the wavehead the dimensionless governing equations are formulated very well in References 3 and 4.

The equations are transformed by Howarth variables as

$$x = x, \qquad y = \int_0^Y \rho \, \mathrm{d}y, \qquad t = t,$$

$$u = u, \qquad v = \left(\rho v + \frac{\partial y}{\partial t} + u \frac{\partial y}{\partial x} \right). \tag{1}$$

Then the governing equations are written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{T}{p} \frac{\mathrm{d}p}{\mathrm{d}x} + p \frac{\partial^2 u}{\partial y^2}, \tag{3}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{p}{Pr}\frac{\partial^2 T}{\partial y^2}+\frac{\gamma}{\gamma-1}\frac{T}{p}\left[\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\frac{p^2}{T}\left(\frac{\partial u}{\partial y}\right)^2\right]. \tag{4}$$

The boundary conditions at the wall ($y=0$) are given by

$$u=1, \qquad v=0, \qquad T_w=\text{const} \quad \text{or} \quad (\partial T/\partial y)_w=\text{const}. \tag{5}$$

At the outer edge ($y\to\infty$) the inviscid solutions are given by Huter *et al.*[1] as

$$U_e=1\pm\frac{2}{\gamma+1}\xi=F(\xi),$$

$$T_e=\frac{1}{\gamma}\left(1\mp\frac{\gamma-1}{\gamma+1}\xi\right)^2=G(\xi),$$

$$P_e=\left(1\mp\frac{\gamma-1}{\gamma+1}\xi\right)^{2\gamma/(\gamma-1)}=H(\xi), \tag{6}$$

where the similarity variable is $\xi=x/t$ and the upper and lower signs are for expansion and compression waves respectively.

The equations are transformed for the initial conditions by using the semisimilarity variables

$$\xi=x/t, \qquad \eta=y/x^{1/2}, \qquad \psi=x^{1/2}U_e f, \qquad g=T/T_e, \qquad u=\partial\psi/\partial y, \qquad v=-\partial\psi/\partial x. \tag{7}$$

This transformation reduces the number of independent variables from three (t, x, y) to two (ξ, η) and the governing equations from partial to ordinary differential equations at the initial point.

The transformed equations are written as

$$m_1 f'''+m_2 ff''+m_3 f'^2+m_4 f'+m_5 g=m_6\left(f'\frac{\partial f'}{\partial\xi}-f''\frac{\partial f}{\partial\xi}\right)+m_7\frac{\partial f'}{\partial\xi}, \tag{8}$$

$$\frac{1}{Pr}m_1 g''+m_2 fg'+m_8 f''^2=m_6\left(f'\frac{\partial g}{\partial\xi}-g'\frac{\partial f}{\partial\xi}\right)+m_7\frac{\partial g}{\partial\xi}. \tag{9}$$

Here the coefficients $m_1, m_2, \ldots, m_8$ are functions of ξ only, as follows

$$m_1=H, \qquad m_2=\frac{1}{2}F+\xi\frac{\partial F}{\partial\xi}, \qquad m_3=-\frac{\partial F}{\partial\xi}, \qquad m_4=\frac{\xi}{F^2}\frac{\partial F}{\partial\xi},$$

$$m_5=-\xi\frac{G}{FH}\frac{\partial H}{\partial\xi}, \qquad m_6=F\xi, \qquad m_7=\mp\xi^2, \qquad m_8=\frac{\gamma-1}{\gamma}\frac{HF^2}{G}, \tag{10}$$

where the upper sign is for the expansion wave and the lower is for the compression wave. The boundary conditions for an isothermal or heat flux wall are

$$y=0: \quad f=0, \quad f'=1/U_e, \quad g=1/T_e \ \text{or} \ g'=g'_w;$$

$$y\to\infty; \quad f'\to 1, \quad g\to 1. \tag{11}$$

The initial conditions are obtained from the reduced governing equations by the substitution $\xi=0$.

NUMERICAL FORMULATION

The order of differential equations is reduced by the introduction of u, v and h, where the symbols u and v are reused for the new definition:

$$f'=u \tag{12}$$

$$u' = v, \tag{13}$$

$$m_1 v' + m_2 f v + m_3 u^2 + m_4 u + m_5 g = m_6\left(u\frac{\partial u}{\partial \xi} - v\frac{\partial f}{\partial \xi}\right) + m_7\frac{\partial u}{\partial \xi}, \tag{14}$$

$$g' = h,. \tag{15}$$

$$\frac{1}{Pr} m_1 h' + m_2 f h + m_8 v^2 = m_6\left(u\frac{\partial g}{\partial \xi} - h\frac{\partial f}{\partial \xi}\right) + m_7\frac{\partial g}{\partial \xi}. \tag{16}$$

The boundary conditions are

$$\eta = 0; \quad f = 0, \quad u = 1/F, \quad g = 1/G \quad \text{or} \quad g' = g'_w;$$

$$\eta \to \infty; \quad u \to 1, \quad g \to 1. \tag{17}$$

Equations (12)–(16) are solved by advancing the solution in the ξ-direction. Discretization in the ξ-direction is done by using incremental values Δ as

$$W^{n+1} = W^n + \Delta W^n, \tag{18}$$

where W represents any variable f, u, v, h, g or ξ.

By using a generalized ξ-differencing formula developed by Orlandi and Ferziger,[5] discretization of the derivatives in the ξ-direction is done as follows:

$$\left[\left(\frac{\partial u}{\partial \xi}\right)^n + \phi\left(\frac{\partial \Delta u}{\partial \xi}\right)\right]\frac{\Delta \xi}{1+\zeta} = \Delta u^n - \frac{\zeta}{1+\zeta}\Delta u^{n-1} + O[(\phi - \zeta - \tfrac{1}{2})\Delta\xi^2 + \Delta\xi^3]. \tag{19}$$

Derivatives of f and g can also be discretized.

Among the various schemes identifiable by the two parameters $|\phi| \leqslant 1$ and $|\zeta| \leqslant 1$, a second-order-accurate three-point backward implicit scheme is selected by letting $\phi = 1$ and $\zeta = \frac{1}{2}$.

Multiplying equation (19) by u and letting $\phi = 1$ and $\zeta = \frac{1}{2}$, the equation becomes

$$\left[\left(u\frac{\partial u}{\partial \xi}\right)^n + \Delta\left(u\frac{\partial u}{\partial \xi}\right)^n\right]\frac{\Delta \xi}{1+\zeta} = u^n \Delta u^n - \frac{\zeta}{1+\zeta} u^n \Delta u^{n-1} + O[(\Delta u^n)^2 + \Delta\xi^3]. \tag{20}$$

This procedure is applied for the ξ-direction derivatives and central differences are used in the η-direction.

The resulting finite difference equations are written as

$$\Delta f_{ij} - \Delta f_{ij-1} - \tfrac{1}{2}\Delta\eta_j(\Delta u_{ij} + \Delta u_{ij-1}) = a_j, \tag{21}$$

$$\Delta u_{ij} - \Delta u_{ij-1} - \tfrac{1}{2}\Delta\eta_j(\Delta v_{ij} + \Delta v_{ij-1}) = b_{j-1}, \tag{22}$$

$$(S_1)_j \Delta v_{ij} + (S_2)_j \Delta v_{ij-1} + (S_3)_j \Delta f_{ij} + (S_4)_j \Delta f_{ij-1} + (S_5)_j \Delta u_{ij} + (S_6)_j \Delta u_{ij-1} + (S_7)_j \Delta g_{ij} + (S_8)_j \Delta g_{ij-1} = c_j, \tag{23}$$

$$\Delta g_{ij} - \Delta g_{ij-1} - \tfrac{1}{2}\Delta\eta_j(\Delta h_{ij} + \Delta h_{ij-1}) = d_{j-1}, \tag{24}$$

$$(P_1)_j \Delta h_{ij} + (P_2)_j \Delta h_{ij-1} + (P_3)_j \Delta f_{ij} + (P_4)_j \Delta f_{ij-1} + (P_5)_j \Delta g_{ij} + (P_6)_j \Delta g_{ij-1} + (P_7)_j \Delta u_{ij} + (P_8)_j \Delta u_{ij-1} + (P_9)_j \Delta v_{ij} + (P_{10})_j \Delta v_{ij-1} = e_j, \tag{25}$$

where a, b, c, d, e, S and P are functions of known status.

The above system of equations (21)–(25) can produce a 5×5 block tridiagonal matrix equation when assembled.

In accordance with the wall thermal boundary conditions, i.e. the temperature or heat flux condition, the five-dimensional vectors Δ_j and $\mathbf{r}_j$ are defined differently for each value of j. For the wall temperature condition the vectors are

$$\Delta_j=\begin{bmatrix}\Delta f_j\\ \Delta u_j\\ \Delta g_j\\ \Delta v_j\\ \Delta h_j\end{bmatrix},\quad \mathbf{r}_1=\begin{bmatrix}0\\ 0\\ \Delta g_w\\ b_1\\ d_1\end{bmatrix},\quad \mathbf{r}_j=\begin{bmatrix}a_j\\ c_j\\ e_j\\ b_j\\ d_j\end{bmatrix},\quad \mathbf{r}_J=\begin{bmatrix}a_J\\ c_J\\ e_J\\ 0\\ 0\end{bmatrix}. \tag{26}$$

For the heat flux condition the vectors are

$$\Delta_j=\begin{bmatrix}\Delta f_j\\ \Delta u_j\\ \Delta h_j\\ \Delta v_j\\ \Delta g_j\end{bmatrix},\quad \mathbf{r}_1=\begin{bmatrix}0\\ 0\\ \Delta h_w\\ b_1\\ d_1\end{bmatrix},\quad \mathbf{r}_j=\begin{bmatrix}a_j\\ c_j\\ e_j\\ b_j\\ d_j\end{bmatrix},\quad \mathbf{r}_J=\begin{bmatrix}a_J\\ c_J\\ e_J\\ 0\\ 0\end{bmatrix}. \tag{27}$$

The coefficient 5×5 matrices are composed of the coefficients of equations (21)–(25). This matrix equation is solved without difficulty by using the well-known block elimination method.[6–8]

RESULTS AND DISCUSSION

The results are presented for isothermal and adiabatic wall conditions with $Pr=0{\cdot}72$ and $\gamma=1{\cdot}4$. The interesting physical quantities are the skin friction, the heat flux and the displacement thickness, which are formulated as

$$\begin{aligned} C_f(Re\,t)^{1/2}&=FH|f''|\xi^{-1/2},\\ q_w(t)^{1/2}&=-HG|g'|\xi^{-1/2},\\ \delta(Re\,t)^{1/2}&=\xi^{-1/2}\int_0^\infty|1-f'|\,\mathrm{d}\eta. \end{aligned} \tag{28}$$

The skin friction is plotted for isothermal and adiabatic walls in Figure 1. The skin friction for the expansion wave increases rapidly behind the wavefront until it approaches a maximum and then slowly decreases. The skin friction for the compression wave increases almost linearly with ξ. Because these flows have the characteristics of an initially developed boundary layer, the skin friction increases rapidly at the wavefront. After the boundary layer is sufficiently developed, it decreases slowly in the decelerated flow and increases linearly in the accelerated flow. Because of the viscosity change due to the wall heat flux, the skin friction on the isothermal wall is larger than on the adiabatic wall for the expansion wave. However, the opposite phenomenon can be seen in the compression wave flow.

The plot of heat flux is given in Figure 2 for the isothermal wall. The heat flux increases rapidly and then decreases slowly as a result of the growing boundary layer thickness in the expansion wave flow. The trend of the wall heat flux curve for the compression wave flow is similar to the skin friction curve, but the former has a higher slope.

These results are compared with the series expansion solutions[4] given for the expansion wave flow. The difference between the two solutions increases gradually with ξ owing to the drawback of series solution. In other words, the numerical error of the series solutions is proportional to ξ^n but that of the present method is proportional to $\Delta\xi^2$.

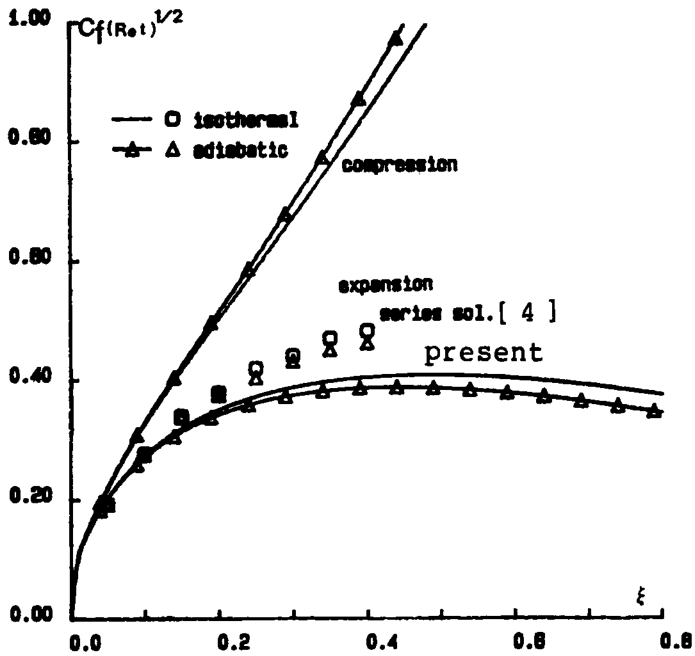

Figure 1. The skin friction

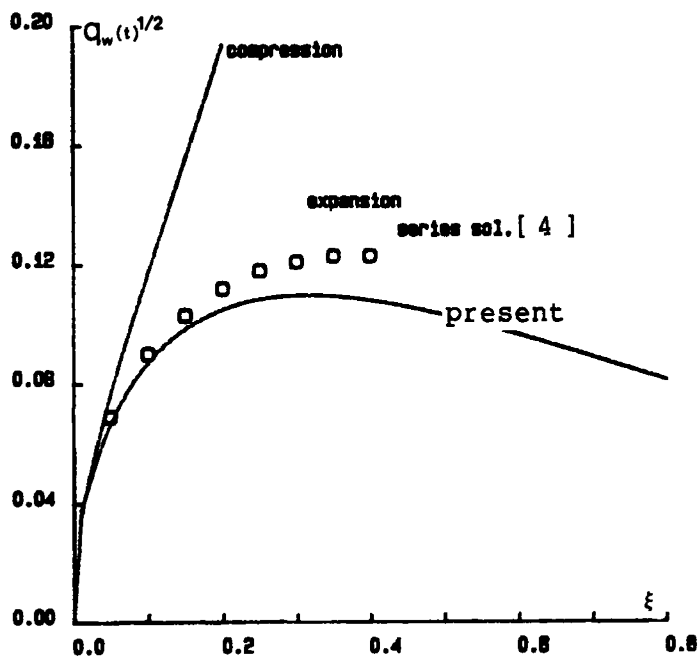

Figure 2. The heat flux

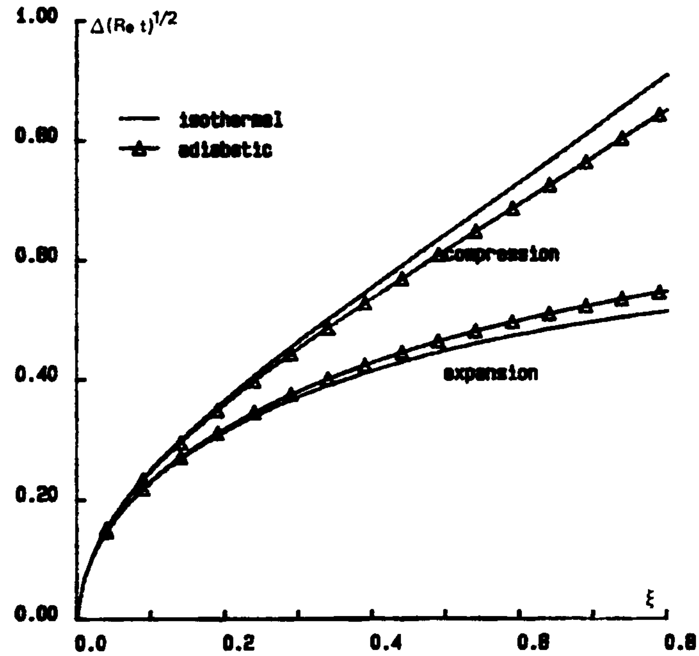

Figure 3. The pseudodisplacement thickness

The pseudodisplacement thickness, where 'pseudo' means that the value is formulated in the co-ordinate system of Howarth transformation, is plotted in Figure 3. The thickness grows rapidly behind the front of the expansion wave but slowly thereafter. In the compression wave it grows almost linearly with increasing ξ. The curves give an idea of how the thickness grows with time at a given point x and how it is distributed in x at a given time. Because of the wall heat flux, the displacement thickness in the isothermal wall flow is smaller than in the expansion wave flow. The opposite phenomenon can be seen in the compression wave flow.

CONCLUSIONS

A numerical method and results are presented for compressible laminar boundary layers excited by an expansion or compression wave with constant wall temperature or constant wall heat flux condition.

A semisimilarity transformation and numerical linearization technique are used to deal with the difficulties caused by the non-linearity, the coupling and the initial condition. The semi-similarity transformation reduces the number of independent variables from three to two and the governing equations from partial to ordinary differential equation at the initial point. Therefore exact numerical initial conditions can be obtained. Because the numerical linearization technique eliminates the non-linear and coupling terms in the governing equations, the solutions can be obtained without iterations.

This technique is much improved compared with series expansion or approximate methods. The numerical error of series solutions is proportional to ξ^n but that of the present method is proportional to $\Delta\xi^2$. The results are compared with the series expansion solutions for expansion wave flow.

APPENDIX: NOMENCLATURE

C_f	coefficient of local skin friction
F	inviscid velocity function
f	dimensionless streamfunction
G	inviscid temperature function
g	dimensionless temperature function
H	inviscid pressure function
p	dimensionless pressure
Pr	Prandtl number ($=0.72$)
q_w	dimensionless wall heat flux
R	gas constant
Re	Reynolds number
T	dimensionless temperature
T_0	temperature in undisturbed region
t	dimensionless time
u, v	dimensionless velocity components in (x, y)-co-ordinates
U_r	characteristic velocity ($=\sqrt{(\gamma R T_0)}$)
ξ, η	dimensionless stream co-ordinate system
γ	ratio of specific heats ($=1.4$)
μ	dimensionless viscosity (proportional to temperature)
ρ	dimensionless density
ψ	dimensionless streamfunction
δ	displacement thickness

subscripts

c	characteristic value
e	external value
w	wall value

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